Journal of Statistical Physics, Vol. 80, Nos. 1/2, 1995

A Discrete Variational Problem Related to Ising Droplets at Low Temperatures

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Received May 11, 1994; final December 8, 1994

We consider a variational problem on the d-dimensional lattice Z^d which has applications in the study of the metastable behavior of the stochastic Ising model. The problem, an isoperimetric one, is to find what is the smallest area a finite subset of Z^d can have restricted to three classes of subsets of Z^d. If ϕ is one of these subsets, we define its volume as the number of points in it and its area as the number of pairs of points in Z^d which are neighbors and such that only one of them belongs to ϕ.

KEY WORDS: Discrete variational problem; Ising model; droplets; metastability.

1. INTRODUCTION

We consider a variational problem which arises from the analysis of the metastable behavior of the finite-volume d-dimensional stochastic Ising model for very low temperatures.[5-9] The problem is to find what is the smallest possible area within certain classes of subsets of Z^d. For each $\phi \subset Z^d$, finite, we define its volume as the cardinality of this set, denoted by $|\phi|$, and its area, denoted by $A^d(\phi)$, as the number of edges with only one endpoint in ϕ,

$$A^d(\phi) = |\{\{x, y\} : x \in \phi,\ y \notin \phi, \text{ with } d(x, y) = 1\}| \tag{1}$$

where $d(x, y) \equiv \sum_{i=1}^{d} |x_i - y_i|$ is the lattice distance. If ϕ is a subset of an i-dimensional subspace S of Z^d, we also define the area of ϕ inside this subspace

$$A^i(\phi) = |\{\{x, y\} : x \in \phi,\ y \in S \backslash \phi \text{ with } d(x, y) = 1\}| \tag{2}$$

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The minimization problem is considered in three classes of subsets of Z^d. The first class is

$$\mathscr{C}(v) = \{\phi \subset Z^d : |\phi| = v\} \tag{3}$$

The other two classes, denoted by $\mathscr{C}^{\eta,+}(v)$ and $\mathscr{C}^{\eta,-}(v)$, are those containing subsets of Z^d which can be obtained from a given parallelepiped η either by addition or by removal of v lattice sites:

$$\mathscr{C}^{\eta,+} = \{\phi \supset \eta : |\phi| = |\eta| + v\}, \qquad \mathscr{C}^{\eta,-} = \{\phi \subset \eta : |\phi| = |\eta| - v\} \tag{4}$$

To our knowledge this discrete isoperimetric problem has not been solved before, though similar questions appear to be natural in the context of combinatorics.[1,2]

As already mentioned, these minimization questions arise in the analysis of metastability for the finite-volume stochastic Ising model at very low temperatures. This model is defined in the torus $\Lambda_N = \{1,..., N\}^d$ with N a large but fixed positive integer and with periodic boundary conditions. The Hamiltonian is given by

$$H(\sigma) = -\frac{1}{2} \sum_{\langle x, y \rangle} \sigma(x)\,\sigma(y) - \frac{h}{2} \sum_{x} \sigma(x)$$

where $\sigma(x) \in \{-1, +1\}$ is the spin at site $x \in \Lambda_N$, the first sum is taken over all pairs of nearest neighbors in Λ_N, the second sum is taken over all sites in Λ_N, and h is the magnetic field, which we assume positive.

Each configuration $\sigma \in \{-1, +1\}^{\Lambda_N}$ of this model defines a subset of Z^d by $\{x: \sigma(x) = +1\}$. No confusion should arise if we use the same notation for a configuration and the subset it defines.

The Hamiltonian can also be written as

$$H(\sigma) = A(\sigma) - h\,|\sigma| + H(\underline{-1})$$

where $\underline{-1}(x) = -1$ for $x \in \Lambda_N$ is the configuration with all spins equal -1, $|\sigma| = |\{x \in \Lambda_N : \sigma(x) = +1\}|$, and $A(\sigma)$ is the number of nearest-neighbor sites with different spins. Therefore the variational problem restricted to the class $\mathscr{C}(v)$ corresponds to looking for the smallest possible energy among the configurations with a given *magnetization*

$$M_\Lambda = \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \sigma(x)$$

There are many ways to introduce a stochastic dynamics in this system. We introduce a Glauber dynamics[4] on which only one spin can flip at a time and does so with Metropolis rate. This version of the stochastic Ising model is the process $\{\sigma_t^\eta\}_{t\geq 0}$ on $\{-1, +1\}^{\Lambda_N}$ with $c(x, \eta)$, the rate with which the spin at site x flips when the current configuration is η, given by

$$c(x, \eta) = \exp - \beta[\Delta_x H(\eta)]^+$$

where

$$\Delta_x H(\eta) = H(\eta^x) - H(\eta) = \eta(x)\left[\sum_{y:\,\langle x, y\rangle} \eta(y) + h\right]$$

with $\eta^x(y) = \eta(y)$ if $x \neq y$ and $\eta^x(x) = -\eta(x)$, the sum being taken over all nearest neighbors of x and for a real number x, $[x]^+ = \max\{0, x\}$. The process $\{\sigma_t^\eta\}_{t\geq 0}$ is reversible with respect to the Gibbs measure given by

$$\mu(\sigma) = (Z_N)^{-1} \exp[-\beta H(\sigma)]$$

$$Z_N = \sum_\sigma \exp[-\beta H(\sigma)]$$

where β is the inverse temperature of the system.

One nice feature of this version is that it may be constructed in a very simple way: at each event of a Poisson point process $\{N(t)\}_{t\geq 0}$ with rate N^d one chooses a site x in Λ_N with uniform distribution and flips its spin with probability $c(x, \eta)$.

In $\{-1, +1\}^{\Lambda_N}$ there is a natural partial order given as follows: $\eta \leq \zeta$ if and only if $\eta(x) \leq \zeta(x)$ for all $x \in \Lambda_N$. This model is *ferromagnetic* or *attractive* in the following sense: if $\eta(x) = \zeta(x) = +1$ and $\eta \leq \zeta$, then $c(x, \eta) \geq c(x, \zeta)$ and if $\eta(x) = \zeta(x) = -1$ and $\eta \leq \zeta$, then $c(x, \eta) \leq c(x, \zeta)$.

For $A \subset \{-1, +1\}^{\Lambda_N}$ define the hitting time of A starting at η,

$$T^\eta(A) = \inf\{t \geq 0: \sigma_t^\eta \in A\}$$

If $A = \{\zeta\}$, $\zeta \in \{-1, +1\}^{\Lambda_N}$, we write, for simplicity, $T^\eta(\zeta)$ instead of $T^\eta(\{\zeta\})$.

One way to describe metastability is via the *pathwise approach.*[3] The metastable behavior at low temperatures manifests itself basically in that the system, for some choices of the initial configuration, may apparently reach equilibrium with respect to a measure which is quite different from the invariant measure for the process before it eventually reaches the true equilibrium measure; moreover, the time on which this transition takes place is completely unpredictable as it converges, properly normalized and as the temperature decreases, to an exponential random variable.[7-9].

To give an example on which the discrete variational problem considered here is needed, consider the problem of stability of *regular droplets* in the three-dimensional finite-volume stochastic Ising model at very low temperatures. The problem is the study of the evolution of this process as it starts from a configuration where all $+1$ spins are inside a single regular cluster. More precisely, we take as initial configurations for the process the set $\mathscr{R}_3$ of configurations such that all spins are -1 except those inside a parallelepiped with sides $l_1(\eta)$, $l_2(\eta)$, and $l_3(\eta)$ with $l_1(\eta) \geqslant l_2(\eta) \geqslant l_3(\eta)$.

Starting in $\eta \in \mathscr{R}_3$, the process will spend a long time, "near" this configuration if the temperature is low, but eventually the droplet, that is, the subset of Λ_N where the $+1$ are, will either "shrink" and the process reaches $\underline{-1}$, the configuration with all spins equal -1, or "grow" and the process reaches $\underline{+1}$, the configuration with all spins $+1$. The decision on whether $\underline{-1}$ or $\underline{+1}$ is reached first is not random for very low temperatures, but is determined by a sharp condition on the sizes of the two smallest sides of the parallelepiped defined by the initial configuration η. Moreover, it is possible to determine the scale of time needed for this decision, that is, the relaxation time for the process.

Write L for the smallest integer larger than $2/h$ and let $\Gamma(h) = 4L - L^2h + Lh - h$. Now, $\Gamma(h)$ is the cost of energy to produce what was called[7] the "proto-critical droplet" in the two-dimensional Ising model starting from the configuration with all spins equal to -1. This corresponds to a configuration with all sins -1 except those inside a rectangle with lengths L and $L-1$ together with an additional site adjacent to one of its larger sides. L is the critical length in the two-dimensional case. As was shown in ref. 7, this is a configuration through which the process will pass with large probability for low temperatures as it moves from $\underline{-1}$ to $\underline{+1}$ and the time it takes for this transition grows like $\exp \beta\Gamma(h)$ as the temperature decreases.

With no loss in generality we may take $\eta \in \mathscr{R}_3$ to be such that

$$\begin{aligned} \eta(x) &= +1 \qquad \text{if} \quad x \in \{1,..., l_1(\eta)\} \times \{1,..., l_2(\eta)\} \times \{1,..., l_3(\eta)\} \\ \eta(x) &= -1 \qquad \text{otherwise} \end{aligned}$$

For a positive integer k define

$$Q_c(k) = \begin{cases} 2k/(kh-2) & \text{if} \quad k > 2/h \\ \infty & \text{otherwise} \end{cases} \tag{5}$$

In ref. 6 the "generic case" for the magnetic field was considered by imposing that it does not assume a countable number of values. There was also the restriction to the more interesting case on which $h < 1$ and the volume of the whole system is large enough. More precisely, define $\mathscr{H} = \{h \in (0, 1): h = 2/l + 2/m$ for two positive integers l and $m\}$. For a given $\eta \in \mathscr{R}_3$ we say that we are in the *standard case* if $h \in (0, 1) \backslash \mathscr{H}$ and $N > l_1(\eta) + 2$.

Under this condition $2/h$ is not an integer and neither is $Q_c(l_3(\eta))$ for any $\eta \in \mathscr{R}_3$. The general case can be considered by taking h slighty smaller or larger so that we are in the standard case and using coupling.

The result for the stability of regular droplets[6] is:

Theorem 1. Let $\eta \in \mathscr{R}_3$ in the standard case. For any $\varepsilon > 0$:

(a) If $l_2(\eta) < Q_c(l_3(\eta))$,

$$\lim_{\beta \to \infty} P(T^\eta(\underline{-1}) < T^\eta(\underline{+1}),\ T^\eta(\underline{-1}) < \exp \beta[E(h) + \varepsilon]) = 1$$

where

$$E(h) = \begin{cases} \Gamma(h) - 2[l_3(\eta) + l_2(\eta)] + l_3(\eta)\, l_2(\eta)\, h & \text{if } \ l_3(\eta) > 2/h \\ h(L-2) & \text{otherwise} \end{cases} \tag{6}$$

(b) If $l_2(\eta) > Q_c(l_3(\eta))$

$$\lim_{\beta \to \infty} P(T^\eta(\underline{+1}) < T^\eta(\underline{-1}),\ T^\eta(\underline{+1}) < \exp \beta[\Gamma(h) + \varepsilon]) = 1$$

It is shown in ref. 6 that each configuration in $\mathscr{R}_3$ defines a "basin of attraction" in the sense hat if the process starts "near" this configuration it goes to $\mathscr{R}_3$ with large probability if β is large. Starting from $\eta \in \mathscr{R}_3$, the process spends most of time at η making many quick trips around this basin of attraction. Eventually, after a very long time at low temperatures, in one of those trips it goes far enough and leaves this basin of attraction. To do this it has to overcome an energy barrier and its height provides a lower bound for the exit time through the following result[7]:

Lemma 1. Let $\mathscr{S}$ be a connected set and $\eta \in \mathscr{S}$ such that $H(\eta) < H(\zeta)$ for all $\zeta \in \mathscr{S} \backslash \{\eta\}$. Then for all $\zeta \in \mathscr{S}$ and $\varepsilon > 0$

$$\lim_{\beta \to \infty} P(\tilde{T}(\zeta) < \exp \beta[H(\zeta) - H(\eta) - \varepsilon]) = 0$$

where $\tilde{T}(\zeta) = \inf\{t \geqslant 0; \tilde{\sigma}_t^\eta = \zeta\}$ and $\{\tilde{\sigma}_t^\eta\}_{t \geqslant 0}$ is the process restricted to $\mathscr{S}$.

We say that a subset $\mathscr{S}$ of $\{-1, +1\}^{\Lambda_N}$ is *connected* if for any two configurations in $\mathscr{S}$ the process can move from one to the other without leaving $\mathscr{S}$.

Therefore the process cannot leave the basin of attraction of $\eta \in \mathscr{R}_3$ before time of the order of $\exp \beta \times$(height of the energy barrier) for low temperatures.

To verify the condition of Lemma 1 that η is a local minimum of energy and to find the height of the energy barrier in the basin of attraction of this configuration, we have to solve the variational problem in $\mathscr{C}^{\eta,+}(v)$ and $\mathscr{C}^{\eta,-}(v)$.

The minimization problem in $\mathscr{C}(v)$ arises in the analysis of the height of the energy barrier to be overcome by the process as it goes from $\underline{-1}$ to $\underline{+1}$.

2. RESULTS

We start with the variational results in $\mathscr{C}^{\eta,+}(v)$ and $\mathscr{C}^{\eta,-}(v)$ which are needed to prove Theorem 1.

Define $L(i)$, $1 \leqslant i \leqslant d$, as the smallest integer larger than $2(i-1)/h$ for $i \geqslant 2$ and $L(1) = 2$. A d-dimensional proto-critical droplet, denoted by $\mathbf{p}_d$, is defined as

$$\mathbf{p}_d = \bigcup_{i=1}^{d} b_i \tag{7}$$

where b_i is an i-dimensional parallelepiped with $i-1$ sides with length $L(i)$ and one with length $L(i)-1$ with $b_i \cap b_j = \varnothing$ if $i \neq j$, and if $x \in b_i$, then x is neighbor to b_j for $j > i$.

Let $\mathscr{R}_d$ be the class of d-dimensional parallelepipeds. If $\eta \in \mathscr{R}_d$, write $l_1(\eta) \geqslant l_2(\eta) \geqslant \cdots \geqslant l_d(\eta)$ for the lengths of its sides.

Let $\eta = \{1,..., l_1(\eta)\} \times \cdots \times \{1,..., l_d(\eta)\} \in \mathscr{R}_d$. Define $\bar{\eta}$ as a configuration obtained from η by the addition of all sites inside a $(d-1)$-dimensional proto-critical droplet in $\{l_1(\eta)+1\} \times \{1,..., l_2(\eta)\} \times \cdots \times \{1,..., l_d(\eta)\}$ and $\underline{\eta}$ as one obtained from η by removing all sites inside $\{l_1(\eta)\} \times \{1,..., l_2(\eta)\} \times \cdots \times \{1,..., l_d(\eta)\}$ except those in a $(d-1)$-dimensional proto-critical droplet.

Theorem 2. Let $\eta \in \mathscr{R}_d$, $d \geqslant 2$, be as above, with $l_d(\eta) \geqslant L(d-1)$. Then:

(a) $H(\bar{\eta}) = \min\{H(\zeta); \zeta \in \mathscr{C}^{\eta,+}(|\mathbf{p}_{d-1}|)\}$.

(b) $H(\underline{\eta}) = \min\{H(\zeta); \zeta \in \mathscr{C}^{\eta,-}(|\mathbf{p}_{d-1}|)\}$.

(c) $H(\zeta) - H(\eta) > 0$ if

$$\zeta \in \left(\bigcup_{v=0}^{|\mathbf{p}_{d-1}|} \mathscr{C}^{\eta,+}(v) \right) \cup \left(\bigcup_{v=0}^{|\mathbf{p}_{d-1}|} \mathscr{C}^{\eta,-}(v) \right)$$

To state the variational result restricted to $\mathscr{C}(v)$ we need some notation.

If $\phi \subset Z^d$, let $\mathscr{E}(\phi)$ be the class of all configurations that can be obtained from ϕ by lattice translations, lattice rotations, and lattice reflections. Write $\partial\phi$ for the *external boundary* of $\phi \subset Z^d$,

$$\partial\phi \equiv \{ y \notin \phi\text{: there exists } x \in \phi \text{ with } d(x, y) = 1\} \tag{8}$$

and $B(\phi)$ for the box

$$B(\phi) = \{1 \leqslant x_i \leqslant l_i(\phi),\ 1 \leqslant i \leqslant d\} \tag{9}$$

where, for $\rho \subset Z^d$, finite

$$\begin{aligned} l_i(\rho) = {} & \max\{ j\text{: } \rho \cap \{x_i = j\} \neq \varnothing\} \\ & - \max\{k\text{: } \rho \cap \{x_i = j\} = \varnothing, \text{ for } j \leqslant k\} \end{aligned} \tag{10}$$

is the length of ρ along direction i.

Let Σ^d be the class of configurations defined as follows: $\varnothing \in \Sigma^d$ if $\phi \neq \varnothing$:

(i) $x = (x_1,..., x_d) \in \phi \Rightarrow x_i \geqslant 1,\ 1 \leqslant i \leqslant d.$

(ii) $\{x_i = 1\} \cap \phi \neq \varnothing,\ 1 \leqslant i \leqslant d.$

(iii) $l_i(\phi) \geqslant l_j(\phi)$ if $i \leqslant j$.

To simplify the notation, we write $\{x_i = k\}$ instead of $\{x = (x_1,..., x_d) \in Z^d\text{: } x_i = k\}$.

A set $\phi \subset Z^d$ is called a *j-dimensional block* if it is a parallelepiped with $d - j$ sides with length 1 and the remaining j sides either all equal or assuming two successive positive integers. That is, a j-dimensional block is a set

$$\phi \in \mathscr{E}(\{x \in Z^d\text{: } 1 \leqslant x_i \leqslant L_i\})$$

with $L_i = 1$ if $i > j$ and $L_i \in \{M, M+1\}$, $1 \leqslant i \leqslant j$, for some positive integer M.

Call $\phi \cap \{x_i = k\}$ a *slice* of ϕ along direction i at position k. Call it an *external slice* on the positive (negative) direction i if it is nonempty but $\phi \cap \{x_i = k+1\} = \varnothing$ ($\phi \cap \{x_i = k-1\} = \varnothing$). Clearly any slice of a block is itself a block. On the other hand, if one adds a slice to a block, the result may not be a block (see Remark 2 below).

For further reference we now organize some simple facts about blocks which will be used later.

Let ϕ be an i-dimensional block in Σ^d which we may take, without loss of generality, as a subset of

$$S_i = \{x_d = 1, \ldots, x_{i+1} = 1\} \tag{11}$$

with $S_d \equiv Z^d$.

Remark 1. The external boundary of a block ϕ, $\partial\phi$, is the union of $2d$ disjoint blocks $b_1, \ldots, b_{2d}$. Each b_k may be obtained by translation of one lattice unit of an external slice toward the outside of ϕ. These blocks are not connected to each other, as, if x and y are points in different blocks, then $d(x, y) > 1$ [the notion of connectivity is the usual one in percolation theory: a set $\mathscr{S} \subset Z^d$ is connected if for any pair of its points, say x and y, there exists a sequence $\{z_i\}_{i=1}^n$, for some positive integer n, in $\mathscr{S}$ with $z_1 = x$, $z_n = y$ and $d(z_i, z_{i+1}) = 1$ for $1 \leqslant i < n$]. Now, $2(d-i)$ of those blocks are obtained by translation of ϕ by one lattice unit along the positive and negative directions $i+1, i+2, \ldots, d$ and are copies of ϕ itself. Write $\overline{\partial\phi}$ for the collection of those $2(d-i)$ blocks. The $2i$ remaining blocks are $(i-1)$-dimensional and are also subsets of S_i, (11). Write $\underline{\partial\phi}$ for the collection of those $2i$ blocks. Each block in $\underline{\partial\phi}$ is equal to one external slice of ϕ on some direction $j \in \{1, 2, \ldots, i\}$ translated by one lattice unit along this direction.

The volumes of the blocks in $\underline{\partial\phi}$ may either all be equal or take two different values. If all sides of ϕ along directions $1, 2, \ldots, i$ have the same length, say $l_j(\phi) = l$, $1 \leqslant j \leqslant i$, for some positive integer l, then all elements of $\underline{\partial\phi}$ are equal [$(i-1)$-dimensional blocks with length l] and are said to be *big*. If the sides of ϕ along directions $1, 2, \ldots, i$ are not equal, say $l_j(\phi) = l+1$ for $1 \leqslant j \leqslant k < i$ and $l_j(\phi) = l$ for $k < j \leqslant i$ for some $k \in \{1, \ldots, i-1\}$ and some positive integer l, the volumes of the elements of $\underline{\partial\phi}$ are $l^{k-1}(l+1)^{i-k}$ or $l^k(l+1)^{i-k-1}$. In the first case we say that the block in $\underline{\partial\phi}$ is *big*.

Remark 2. Let ϕ be an i-dimensional block and b be a *big* block in $\underline{\partial\phi}$. Then $\phi \cup b$ is a block.

Remark 3. If ϕ and ψ are different blocks with $\phi \subset \psi$, then at least one of the *big* blocks in $\underline{\partial\phi}$ is contained in ψ.

Remark 4. If ϕ and ψ are i-dimensional blocks with volumes a_1 and a_2, respectively, with $a_1 \leqslant a_2$, then one can find $b \in \mathscr{E}(\psi)$ such that $\phi \subset b$.

Let $A_i = \{a\colon$ there exists an i-dimensional block with volume $a\}$ for $1 \leqslant i \leqslant d$ and for any positive integer v

$$\underline{v}^i = \max\{a \in A_i\colon a \leqslant v\}$$

Define $\mathbf{b}_i(v)$ as the i-dimensional block in Σ^d with volume $\underline{v}^i$. Then $\mathbf{b}_i(v)$ is the largest i-dimensional block with volume not larger than v. We may construct $\mathbf{b}_i(v)$ as follows

If $v > 1$, let $L_i(v)$ and $M_i(v)$ be given by

$$L_i(v) = \min\{l \in N: l^i \geqslant v\} \tag{12}$$

and

$$M_i(v) = \max\{m \in \{0, 1,..., i\} : L_i^m(v)[L_i(v) - 1]^{i-m} \leqslant v\} \tag{13}$$

Set $\mathbf{b}_i(0) = \varnothing$, $\mathbf{b}_i(1) = \{x_j = 1, 1 \leqslant j \leqslant d\}$, and, if $v \geqslant 2$,

$$\mathbf{b}_i(v) = \{1 \leqslant x_j \leqslant l_j; \text{for } 1 \leqslant j \leqslant d\} \tag{14}$$

with $l_j = L_i(v)$ for $1 \leqslant j \leqslant M_i(v)$, $l_j = L_i(v) - 1$ for $M_i(v) < j \leqslant i$, and $l_j = 1$ for $i < j \leqslant d$.

For any positive integer v, let $\{v_i\}_{1 \leqslant i \leqslant d}$ be defined by

$$v_d = |\mathbf{b}_d(v)| \qquad \text{and} \qquad v_i = \left|\mathbf{b}_i\left(v - \sum_{j>i} v_j\right)\right| \tag{15}$$

Then

$$v_1 = \left|\mathbf{b}_1\left(v - \sum_{j>1} v_i\right)\right| = v - \sum_{j>1} v_i \tag{16}$$

The second equality holds since A_1 is the set of natural numbers.

We now define $\mathscr{B}^d(v)$ as the set of configurations which, having volume v, resemble as much as possible a d-dimensional cube and, as we show here, have the smallest possible area. A configuration ξ belongs to $\mathscr{B}^d(v)$ if $|\xi| = v$ and it has a decomposition

$$\xi = \bigcup_{i=1}^{d} \xi_i \tag{17}$$

where each ξ_i, $1 \leqslant i \leqslant d$, is an i-dimensional block with volume v_i [as in (15) above]; moreover, we impose the condition that those blocks are disjoint and that each low-dimensional block is attached to an external slice of the larger-dimensional ones. More precisely, $\{\xi_i\}_{i=1}^{d}$ must satisfy the following:

Condition a: ξ_i is an i-dimensional block with volume v_i,

$$\xi_i \in \mathscr{E}(\mathbf{b}_i(v_i)), \qquad v_i \text{ as in (15) above}$$

Condition b: The decomposition is disjoint

$$\xi_i \cap \xi_j = \varnothing \qquad \text{if} \quad i \neq j$$

Condition c: Each ξ_i, $i<d$, is inside the external boundary of all larger-dimensional blocks:

$$\text{for all pairs} \quad 1 \leqslant i < j \leqslant d \qquad \text{we have} \quad \xi_i \subset \partial\xi_j$$

To prove that these conditions can be satisfied, we give a construction of an element of ξ of $\mathscr{B}^d(v)$ for $v>1$ and $d>1$. If $v_{d-1}=0$, we may take $\xi \in \mathscr{E}(\mathbf{b}_d(v))$. Otherwise take $\xi_d \in \mathscr{E}(\mathbf{b}_d(v))$ and ξ_{d-1} chosen in $\mathscr{E}(\mathbf{b}_{d-1}(v-v_d))$ as a subset of one, say b^d, of the *big* blocks in $\underline{\partial\xi_d}$ (*big* blocks as defined after Remark 1 above). By Remark 2 and the definition of v_d we have $|\xi_{d-1}| = v_{d-1} < |b^d|$ and therefore by Remarks 3 and 4 we can choose $\xi_{d-1} \subset b^d$ and have one of the *big* blocks in $\underline{\partial\xi_{d-1}}$, say b^{d-1}, also included in b^d. If $v_{d-2}=0$, we take $\xi = \xi_d \cup \xi_{d-1}$. Otherwise take $\xi_{d-2} \in \mathscr{E}(\mathbf{b}(v-v_d-v_{d-1}))$ as a subset of b^{d-1} with $\xi_{d-2} \subset \partial\xi_{d-1}$ and $\xi_{d-2} \subset \partial\xi_d$. This construction goes on until either we reach $v_j=0$ for $1<j<d$ or ξ_1 is obtained.

Write $\mathscr{B}^d = \bigcup_{v=0}^{\infty} \mathscr{B}^d(v)$.

We distinguish a subset $\widetilde{\mathscr{B}}^d$ of $\mathscr{B}^d$ of what we call *canonical elements* or *canonical configurations* as those which satisfy the additional condition that each *protuberance* $\bigcup_{i \leqslant k} \xi_i$ attached to the block ξ_{k+1} is "almost a block." More precisely, elements of $\widetilde{\mathscr{B}}^d$ must satisfy Conditions a–c above and also the following:

Condition d: $B(\bigcup_{i \leqslant k} \xi_i)$ is a k-dimensional block for $1 \leqslant k \leqslant d$.

An example of a configuration in $\widetilde{\mathscr{B}}^2$ is the proto-critical droplet, that is, a rectangle with sides L and $L-1$, where L is the smallest integer larger than $2/h$, with a single extra site neighbor to one of the largest sides. A configuration in $\mathscr{B}^2 \backslash \widetilde{\mathscr{B}}^2$ is the rectangle with sides with lengths L and $L-1$, as before, with the additional site attached to one of the shortest sides.

The construction sketched above for elements of $\mathscr{B}^d$ in fact only produces elements of $\widetilde{\mathscr{B}}^d$.

Remark 5. In $\widetilde{\mathscr{B}}^d$ each *protuberance* $\bigcup_{i \leqslant k} \xi_i$, $1 \leqslant k \leqslant d$, is a subset of one of the *big* blocks in $\underline{\partial\xi_j}$ for $j \geqslant k+1$ and the inclusion is strict.

Areas of elements of $\mathscr{B}^d$ are simple to compute. Take $\xi \in \mathscr{B}^d(v)$ with $\xi_i = \bigcup_{i=1}^{d} \xi_i$, $\xi_i \in \mathscr{E}(\mathbf{b}_i(v_i))$ as in (17).

Then

$$A^d(\xi) = \sum_{i=1}^{d} A^i(\xi_i) \tag{18}$$

where $A^i(\xi_i)$ is the area of the block ξ_i within the corresponding i-dimensional subspace of Z^d.

Write a_v^d for the smallest possible area in the class of configurations in Z^d with volume v,

$$a_v^d = \min\{A^d(\phi); \phi \subset \mathscr{C}(v)\} \tag{19}$$

The minimization result in $\mathscr{C}(v)$ is as follows.

Theorem 3. If $\xi \in \mathscr{B}^d(v)$, for some $d \geqslant 1$ and $v \geqslant 1$, then

$$A^d(\xi) = a_v^d$$

This result identifies a class of configurations with volume v that we can use to compute a_v^d. It is not true, however, that all subsets of Z^d with volume v and area a_v^d belong to $\mathscr{B}^d(v)$. A simple example in $d = 2$ of a set with smallest area not in $\mathscr{B}^2$ is obtained from a square of length larger than or equal to 3 by removal of two points at the corners on the same diagonal.

The two-dimensional version of this result was used in ref. 7 to verify that the energy barrier between $\underline{-1}$ and $\underline{+1}$ in this case is $\Gamma(h)$ as defined above.

3. PROOFS OF THEOREMS 2 AND 3

We prove first Theorem 3. The proof of Theorem 2 is quite simple once we have the results needed to prove Theorem 3.

Rather than working with *area of a configuration*, it is simpler and sufficient for our purposes to introduce the auxiliary notion of *projected area of a configuration* ϕ, denoted by $PA^d(\phi)$. We define it as twice the number of lines in Z^d which intersect the given configuration. With this definition $PA^d(\phi)$ is equal to $A^d(\phi)$ for a class of configurations which includes $\mathscr{B}^d(v)$. More precisely, if l is a line in Z^d, that is,

$$l = \{x \in Z^d: x = x_0 + ke_i; k \in Z\}$$

where e_i is the unit vector along direction i, $1 \leqslant i \leqslant d$, $x_0 \in Z^d$, and if $\mathscr{L}$ is the set of all lines of Z^d, we define

$$PA^d(\phi) = 2\,|\{l \in \mathscr{L}: l \cap \phi \neq \varnothing\}| \tag{20}$$

If ϕ belongs to a j-dimensional subspace S of Z^d, we also define its projected area within this subspace by considering only lines in S,

$$PA^j(\phi) = 2\,|\{l \in \mathscr{L} : l \subset S;\, l \cap \phi \neq \varnothing\}| \tag{21}$$

It is simple to verify the following result.

Lemma 2. We have:

(a) $PA^d(\xi) \leqslant A^d(\xi)$ for all $\xi \subset Z^d$.

(b) $PA^d(\xi) = A^d(\xi)$ if $\xi \in \mathscr{B}^d$.

(c) $PA^d(\xi) = PA^d(\phi)$ if both ξ and ϕ belong to $\mathscr{B}^d(v)$.

For $d \geqslant 1$ and $v \geqslant 0$ let

$$p_v^d = \min\{PA^d(\xi) : \xi \subset Z^d,\ |\xi| = v\} \tag{22}$$

Theorem 3 follows from the next result.

Lemma 3. If $\xi \in \mathscr{B}^d(v)$, for some $d \geqslant 1$ and $v \geqslant 0$, then

$$PA^d(\xi) = p_v^d$$

That Lemma 3 implies the theorem is a consequence of Lemma 2: if we assume Lemma 3 and $\xi \in \mathscr{B}^d(v)$, then $p_v^d = PA^d(\xi) \leqslant PA^d(\phi)$ for any $\phi \in Z^d$, $|\phi| = v$; by parts (a) and (b) of Lemma 2, $PA^d(\phi) \leqslant A^d(\phi)$ and $PA^d(\xi) = A^d(\xi)$, so that $A^d(\xi) \leqslant A^d(\phi)$ for any $\phi \subset Z^d$, $|\phi| = v$ and the theorem is true.

We prove Lemma 3 using induction in the dimension d. Let $P(d)$ be the property that Lemma 3 is true for dimension d, that is,

$P(d) \equiv$ "For lattice dimension d we have

$p_v^d = PA^d(\xi)$ for all $\xi \in \mathscr{B}^d(v)$ and $v \geqslant 1$"

In dimension $d = 1$ this property is trivial and $\mathscr{B}^1(v)$ corresponds to the set of all intervals of length v in Z.

We now prove that $P(d-1)$ implies $P(d)$ for $d \geqslant 2$. To do this we start with an arbitrary initial configuration in Z^d and use $P(d-1)$ to modify it step by step into a configuration in $\mathscr{B}^d$ with the same volume but with smaller or equal projected area. From this the validity of $P(d)$ follows. We use successive Greek letters to denote the configurations in each step of this process.

Let $\alpha \subset \Sigma^d$, $|\alpha| = v$. As mentioned before, this is the most general case, as any configuration in Z^d has an equivalent one in Σ^d. Write $a_i = |\alpha \cap \{x_1 = i\}|$ for the volume of α in the i-th slice across direction 1.

If $A=\max\{a_i;\ 1\leqslant i\leqslant l_1(\alpha)\}$ with $l_i(\alpha)$ given by (10) and $\mathscr{L}^1\subset\mathscr{L}$ is the set of lines in Z^d which are perpendicular to direction 1, then

$$
\begin{aligned}
PA^d(\alpha) &= 2\,|\{l\in\mathscr{L}\backslash\mathscr{L}^1 : l\cap\alpha\neq\varnothing\}| + \sum_{i=1}^{l_1(\alpha)} PA^{d-1}(\alpha\cap\{x_1=i\}) \\
&\geqslant 2A + \sum_{i=1}^{l_1(\alpha)} PA^{d-1}(\alpha\cap\{x_1=i\})
\end{aligned}
\tag{23}
$$

since the number of lines along direction 1 intersecting α cannot be smaller than A and if $l\in\mathscr{L}^1$, then $l\subset\{x_1=i\}$ for some i.

By (22) we have

$$
PA^d(\alpha)\geqslant 2A+\sum_{i=1}^{l_1(\alpha)} p_{a_i}^{d-1} \tag{24}
$$

Now we verify that if $P(d-1)$ is true, then the right-hand side of (24) actually corresponds to the projected area of a subset of Z^d. To do this we use the following result.

Lemma 4. If $\xi\in\widetilde{\mathscr{B}}^d(v)$ [set of canonical elements of $\mathscr{B}^d(v)$] for some $d\geqslant 1$ and $v\geqslant 1$, then there exists $\eta\in\widetilde{\mathscr{B}}^d(v+1)$ with $\xi\subset\eta$.

Proof. Let $\{v_i\}_{i=1}^d$ and $\{(v+1)_i\}_{i=1}^d$ be the decompositions of v and $v+1$, respectively, as in (15), and let $I=\max\{i\colon (v+1)_i\neq v_i\}\geqslant 1$. If $I=1$, $(v+1)_I=v_I+1$. If $I>1$ and $L\equiv L_I(\sum_{i=1}^I (v+1)_i)$ and $M\equiv M_I(\sum_{i=1}^I (v+1)_i)$, with $L_I(\sum_{i=1}^I (v+1)_i)$ and $M_I(\sum_{i=1}^I (v+1)_i)$ as defined in (12), we have $(v+1)_i=0$ for $i<I$ as $(v+1)_I=L^M(L-1)^{I-M}$, $\sum_{i=1}^I (v+1)_i=\sum_{i=1}^I v_i+1$, and $\sum_{i=1}^I v_i<L^M(L-1)^{I-M}$ [the first equality is the definition of $(v+1)_I$, the second holds because $v_i=(v+1)_i$ for $i>I$, and the third follows from the definition of I]. Therefore

$$
(v+1)_I=\sum_{i\leqslant I} v_i+1 \tag{25}
$$

Consider an arbitrary $\xi=\bigcup_{i=1}^d \xi_i\in\widetilde{\mathscr{B}}^d(v)$ with $\{\xi_i\}_{i=1}^d$ as in (17). We now construct $\eta=\bigcup_{i=1}^d \eta_i\in\widetilde{\mathscr{B}}^d(v+1)$ with $\xi\subset\eta$.

If $\bigcup_{i=1}^I \xi_i=\varnothing$, take $\eta_i=\xi_i$ for $i>I$ and $\eta_I=\{x\}$ with x chosen in a *big* block of $\partial\xi_j$, for $j\leqslant I$ (*big* blocks as defined after Remark 1). That there exists such an x follows from Remark 5 above.

Now suppose $|\bigcup_{i=1}^I \xi_i|=c>0$. If $I<d$, set $\eta_i=\xi_i$ for $i>I$. Take $\eta_I\in\mathscr{E}(\mathbf{b}_I(c+1))$ such that $\eta_I\supset\bigcup_{i=1}^I \xi_i$ and $\eta_k=\varnothing$ for $k<I$. Note that $|\eta_I|=|\mathbf{b}_I(c+1)|=(v+1)_I=|\bigcup_{i=1}^I \xi_i|+1$ by (25). ∎

Let $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_{l_1(\alpha)}$ be an ordering of the numbers $a_1, a_2, \ldots, a_{l_1(\alpha)}$. For each A_i, $1 \leqslant i \leqslant l_1(\alpha)$, find a $\beta_i \subset \{x_1 = i\}$ in $\widetilde{\mathscr{B}^{d-1}}(A_i)$ so that if l is a line along direction 1 and $l \cap \beta_i \neq \varnothing$, then $l \cap \beta_j \neq \varnothing$ for all $1 \leqslant j \leqslant i \leqslant l_1(\alpha)$, so that each β_i is smaller than β_j if $1 \leqslant j < i \leqslant l_1(\alpha)$. This can be done by Lemma 4. In this case we say that $\{\beta_i\}_{i=1}^{l_1(\beta)}$ is a nonincreasing sequence of elements in $\widetilde{\mathscr{B}^{d-1}}$. Note that we could have $a_i = 0$ for some $1 < i < l_1(\alpha)$ (if, for instance, α is disconnected) and in this case we would have $l_1(\beta) < l_2(\alpha)$.

Then $\beta = \bigcup_{i=1}^{l_1(\alpha)} \beta_i$ satisfies

$$PA^d(\alpha) \geqslant 2A + \sum_{i=1}^{l_1(\alpha)} p_{a_i}^{d-1} = PA^d(\beta) \tag{26}$$

Configuration β is the union of $l_1(\beta) \leqslant l_1(\alpha)$ $(d-1)$-dimensional configurations (slices of β), which may be different.

We now verify that if two slices are smaller than the first and larger one, β_1, the best (to minimize the area) is to enlarge one as much as possible at the expense of the other with the restriction that both should remain smaller than β_1. This restriction ensures that the number of lines along direction 1 intersecting the configuration remains equal to A. The following results are used to prove this. They establish properties in any lattice dimension $n \geqslant 1$ provided $P(n)$ holds. We will apply these results in the induction argument with $n = d-1$.

Lemma 5. Let $P(n)$ be true and ξ_1 and ξ_2 be two arbitrary configurations in Z^n, $n \geqslant 1$. Then

$$PA^n(\xi_1) + PA^n(\xi_2) \geqslant PA^n(\eta_1) + PA^n(\eta_2) \tag{27}$$

for any η_1 and η_2 such that $\eta_1 \in \mathscr{B}^n(|\xi_1 \cap \xi_2|)$ and $\eta_2 \in \mathscr{B}^n(|\xi_1 \cup \xi_2|)$.

Proof. As we assume that $P(n)$ holds, it is enough to prove (27) with $\eta_1 = \xi_1 \cap \xi_2$ and $\eta_2 = \xi_1 \cup \xi_2$.

Let l be a line in Z^n. First consider the case in which $l \cap \xi_1 \cap \xi_2 = \varnothing$. If $l \cap (\xi_1 \cup \xi_2)$ is also empty, this line does not contribute to any of the projected areas in the inequality. If $l \cap (\xi_1 \cup \xi_2)$ is not empty, this line contributes to at least one of the projected areas in the left-hand side, while it does the same only for the second term in the right-hand side.

If $l \cap \xi_1 \cap \xi_2$ is not empty, then l intersects ξ_1, ξ_2 and $\xi_1 \cup \xi_2$, therefore contributing to all terms in the inequality (27). ∎

Lemma 6. Let $P(n)$ be true, ξ and ϕ be elements of $\widetilde{\mathscr{B}^n} \cap \Sigma^n$ such that

$$\varnothing \neq \xi \subset \phi$$

and b be an n-dimensional block satisfying $B(\phi) \subset b$, $B(\phi) \neq b$ [$B(\phi)$ as in (9)]. Then there exist σ and τ in $\widetilde{\mathscr{B}}^n \cap \Sigma^n$ satisfying

$$|\sigma| < |\xi|$$
$$\tau \subset b$$
$$|\xi| + |\phi| = |\sigma| + |\tau|$$
$$PA^n(\xi) + PA^n(\phi) \geqslant PA^n(\sigma) + PA^n(\tau)$$

Proof. As $B(\phi)$ is smaller than b, let

$$J = \min\{i: l_i(\phi) < l_i(b)\} \in \{1, ..., n\}$$

Define $\hat{\phi} = \{x \in Z^n; z - x + e_J \in \phi\}$, where $z = (l_1(\phi) + 1, ..., l_n(\phi) + 1)$ and e_J is the unit vector on direction J. This is the configuration obtained from ϕ by inversion on all lattice directions inside $B(\phi)$ followed by a translation of one lattice unit along direction J.

Apply Lemma 5 with $\xi_1 = \hat{\phi}$ and $\xi_2 = \xi$ and take $\eta_1 = \sigma$ and $\eta_2 = \tau$. Then $|\sigma| = |\hat{\phi} \cap \xi| < |\xi|$ because the point with all coordinates equal to one is in ξ (as $\varnothing \neq \xi \in \Sigma^n$) but does not belong to $\hat{\phi}$ [as it would imply that the point with coordinates $x_i = l_i(\phi)$, $i \neq J$, and $x_J = l_J(\phi) + 1$ belongs to ϕ]. It is also clear that $\hat{\phi} \cup \xi$ cannot have more than one additional slice on direction J since $\hat{\phi}$ moves only one lattice unit in that direction. As $\tau \in \widetilde{\mathscr{B}}^n(|\hat{\phi} \cup \xi|)$, the same is true for it, with $\tau \subset b$. ∎

Lemma 7. Let $P(n)$ be true, ξ and ϕ be elements of $\widetilde{\mathscr{B}}^n \cap \Sigma^n$ such that

$$\varnothing \neq \xi \subset \phi$$

and b be an n-dimensional block satisfying $\phi \neq B(\phi) = b$. Then there exist σ and τ in $\widetilde{\mathscr{B}}^n \cap \Sigma^n$ satisfying

$$|\sigma| < |\xi|$$
$$\tau \subset b$$
$$|\xi| + |\phi| = |\sigma| + |\tau|$$
$$PA^n(\xi) + PA^n(\phi) \geqslant PA^n(\sigma) + PA^n(\tau)$$

Remark. This result (and its proof) is very similar to Lemma 6. The difference is that, as $B(\phi) = b$, if we shift it by one lattice unit and apply Lemma 5 as done in the previous lemma, the condition $\tau \subset b$ will not be true.

Proof. Define $\check{\phi} = \{x \in Z^n;\ z - x \in \phi\}$, where $z = (l_1(\phi) + 1, \ldots, l_n(\phi) + 1)$, for the configuration obtained from ϕ by inversion on all lattice directions.

Apply Lemma 5 with $\xi_1 = \check{\phi}$ and $\xi_2 = \xi$ and take $\eta_1 = \sigma$ and $\eta_2 = \tau$. Again $|\sigma| = |\check{\phi} \cap \xi| < |\xi|$ because the point with all coordinates equal to one is in ξ (as $\varnothing \neq \xi \in \Sigma^n$), but does not belong to $\check{\phi}$ [as it would imply that the point with coordinates $x_i = l_i(\phi)$, $1 \leqslant i \leqslant d$, belongs to η, contradicting the hypothesis that $\phi \neq B(\phi)$]. It is also clear that $\check{\phi} \cup \xi$ cannot be larger than b and the same is also true for $\tau \in \widetilde{\mathscr{B}}^n(|\check{\phi} \cup \xi|)$. ∎

We now apply these results to

$$\beta = \bigcup_{i=1}^{l_1(\beta)} \beta_i$$

as in (26), and write $\beta_i = \bigcup_{j=1}^{d-1} \beta_{i,j}$, $\{\beta_{i,j}\}_{j=1}^{d-1}$ being their decomposition in blocks.

Let us say that β_j is *large* if $\beta_{j,d-1}$ is equal to $\beta_{1,d-1}$ translated to $\{x_1 = j\}$ along direction 1 and that it is *small* otherwise.

A β_j small can be of two types:

1. $B(\beta_j)$ is smaller than $\beta_{1,d-1}$, that is, $B(\beta_j) \subset \rho$, $B(\beta_j) \neq \rho$, for ρ equal to the translation of $\beta_{1,d-1}$ to the subspace $\{x_1 = j\}$.

2. $B(\beta_j)$ is equal to the translation of $\beta_{1,d-1}$ to $\{x_1 = j\}$.

Suppose β is such that there is a β_j, $1 \leqslant j < l_1(\beta)$, which is small of type 1. In this case we have at least two slices, β_j and $\beta_{l_1(\beta)}$, which are small of type 1. Apply Lemma 6 with $n = d - 1$, ξ equal to $\beta_{l_1(\beta)}$ translated along direction 1 to $\{x_1 = 1\}$, ϕ equal to β_j translated to $\{x_1 = 1\}$, and b equal to $\beta_{1,d-1}$. Let γ be obtained by replacing β_j with τ and $\beta_{l_1(\beta)}$ by σ. More precisely,

$$\gamma = \bigcup_{i=1}^{l_1(\beta)} \gamma_i, \qquad \gamma_i \in \widetilde{\mathscr{B}^{d-1}}$$

with

$$\gamma_i = \beta_i \qquad \text{if} \quad i \notin \{j, l_1(\beta)\}$$

$\gamma_j = \{x : x - (j-1)\, e_1 \in \tau\}$ is the translation of τ to $\{x_1 = j\}$, and

$$\gamma_{l_1(\beta)} = \{x : x - (l_1(\beta) - 1)\, e_1 \in \sigma\}$$

is the translation of σ along the positive direction 1 to get a configuration in $\{x_1 = l_1(\beta)\}$.

The number of lines of Z^d along direction 1 which intersects γ is still equal to A and Lemma 5 implies that the modification on slices j and $l_1(\beta)$ does not increase the projected area. Therefore

$$PA^d(\alpha) \geqslant PA^d(\beta) \geqslant PA^d(\gamma) \tag{28}$$

The procedure used to go from β to γ can be repeated as long as there is a β_j small of type 1 for some $1 \leqslant j < l_1(\beta)$. As it always removes points from the last slice, $\beta_{l_1(\beta)}$, it is possible that eventually this slice is emptied. In this case, the procedure starts again with the new configuration which is shorter along direction 1 and with removals now occurring at the current last slice along this direction.

Let

$$\delta = \bigcup_{i=1}^{l_1(\delta)} \delta_i, \qquad l_1(\delta) \leqslant l_1(\beta)$$

be the final configuration on which this procedure can no longer be applied because no δ_i is small of type 1 for $1 \leqslant i < l_1(\delta)$.

Suppose that there exists a δ_i, $1 \leqslant i < l_1(\delta)$, which is small of type 2 and therefore there exist at least two slices, δ_i and $\delta_{l_1(\delta)}$, with this property. Apply Lemma 7 with $n = d-1$, ξ equal to $\delta_{l_1(\delta)}$ translated along direction 1 to $\{x_1 = 1\}$, ϕ equal to δ_i translated to $\{x_1 = 1\}$, and b equal to $\delta_{1,d-1}$. Let ε be the configuration obtained by replacing δ_i with τ and $\delta_{l_1(\delta)}$ by σ given by Lemma 7. As before we have

$$PA^d(\gamma) \geqslant PA^d(\delta) \geqslant PA^d(\varepsilon) \tag{29}$$

Apply this procedure as many times as possible. Eventually it can no longer be applied because all slices are large (except, possibly the last one). Thus if we call

$$\zeta = \bigcup_{j=1}^{l_1(\zeta)} \zeta_j$$

this final subset of Z^d, we have that $\zeta_{j,d-1}$, $1 \leqslant j < l_1(\zeta)$, are equal up to a translation along direction 1.

Let $K \in \{2,..., d\}$ be such that

$$l_K(\zeta) = \min\{l_i(\zeta) : i \in \{2,..., d\}\} = \min\{l_i(\zeta_1) : i \in \{2,..., d\}\}$$

Suppose $B(\zeta)$ is not a block. We apply to ζ one of two transformations.

Transformation 1. $B(\zeta)$ is not a block and $l_1(\zeta) \leqslant l_K(\zeta) - 1$. Then

$$a \equiv |\zeta \cap \{x_K = l_K(\zeta)\}| \leqslant l_1(\zeta) \prod_{i \neq 1, K} l_i(\zeta)$$

$$\leqslant [l_K(\zeta) - 1] \prod_{i \neq 1, K} l_i(\zeta) \leqslant |\zeta \cap \{x_1 = 1\}| \equiv b$$

In this case define a new configuration η obtained from ζ by excluding the points in $\zeta \cap \{x_K = l_K(\zeta)\}$ and adding them to the $(d-1)$-dimensional subspace $\{x_1 = l_1(\zeta) + 1\}$ such that

$$\eta_{l_1(\eta)+1} \equiv \eta \cap \{x_1 = l_1(\zeta) + 1\} \in \mathscr{B}^{d-1}(|\zeta \cap \{x_K = l_K(\zeta)\}|)$$

and every line along direction 1 intersecting $\eta \cap \{x_1 = l_1(\eta) + 1\}$ also intersects $\eta \cap \{x_1 = 1\}$.

This last condition can be satisfied as a consequence of the inequality between a and b above and Lemma 4. Note that

$$\eta_j = \zeta_j \backslash \{x_K = l_K(\zeta)\}$$

is a $(d-1)$-dimensional block. Therefore η is also a nonincreasing sequence of elements in $\widetilde{\mathscr{B}}^d \cap \Sigma^d$ with $l_1(\eta) = l_1(\zeta) + 1$ and

$$PA^d(\varepsilon) \geqslant PA^d(\zeta) \geqslant PA^d(\eta)$$

By repeating this procedure if necessary we eventually reach a configuration with the appropriate length on direction 1.

Transformation 2. Consider now the case on which $B(\zeta)$ is not a block and $l_1(\zeta) \geqslant l_2(\zeta) + 1 \geqslant l_K(\zeta) + 1$. In this case we remove points on $\zeta \cap \{x_1 = l_1(\zeta)\}$ and add them to the subspace $\{x_K = l_K(\zeta) + 1\}$.

As all lines along direction 1 which intersect $\zeta_{l_1(\zeta)}$ also intersect ζ_1, we have

$$PA^d(\zeta) = PA^d(\zeta \backslash \zeta_{l_1(\zeta)}) + PA^{d-1}(\zeta_{l_1(\zeta)}) \tag{30}$$

We also have

$$(\zeta \backslash \zeta_{l_1(\zeta)}) \cap \{x_K = 1\} \supset \{1 \leqslant x_i \leqslant b_i\} \tag{31}$$

with $b_1 = l_1(\zeta) - 1$, $b_K = 1$, and $b_i = l_i(\zeta)$ for $i \notin \{1, K\}$. Moreover, as we assume $l_1(\zeta) \geqslant l_2(\zeta) + 1$,

$$\{1 \leqslant x_i \leqslant b_i\} \supset \{1 \leqslant x_i \leqslant c_i\} \tag{32}$$

with $c_1 = l_2(\zeta)$, $c_K = 1$, and $c_i = l_i(\zeta)$. The right-hand side of (32) is a $(d-1)$-dimensional block (therefore an element of $\widetilde{\mathscr{B}^{d-1}}$) with volume that is larger than or equal to $|\zeta_{l_1(\zeta)}|$. Thus, by Lemma 4, we can find $\omega \in \mathscr{E}(\psi)$ such that (a) $w \in \{x_K = l_K(\zeta) + 1\}$, (b) $\psi \in \widetilde{\mathscr{B}^{d-1}}(|\zeta_{l_1(\zeta)}|)$, and (c) all lines along direction K intersecting ω also intersect $\zeta \backslash \zeta_{l_1(\zeta)}$. If we define $\theta = (\zeta \backslash \zeta_{l_1(\zeta)}) \cup \omega$, we have

$$PA^d(\zeta) = PA^d(\zeta \backslash \zeta_{l_1(\zeta)}) + PA^{d-1}(\omega) = PA^d(\theta) \tag{33}$$

The first equality in (33) is true by (30), (a) and (b) of the definition of ω, and part (c) of Lemma 2. The second equality holds by (c) [as in (23) above, with $\mathscr{L}^1$ replaced by $\mathscr{L}^K$, the set of lines of Z^d perpendicular to direction K]. Now organize the slices of θ which satisfy $l_1(\zeta) = l_1(\theta) + 1$ as done in transforming α to ζ and repeat the previous arguments until the appropriate length along direction 1 is obtained.

Therefore if $B(\zeta)$ is not a block, it can be modified as described above into a configuration, say ι, so that $\{1 \leqslant x_i \leqslant l_i(\iota),\ 1 \leqslant i \leqslant d\}$ is a d-dimensional block. Moreover, ι is almost a block in the sense that it is a block with at most two faces which are eroded. If only one face is eroded, then the theorem is proven. Assume that this is not the case, that is, there exist $Q \in \{2, ..., d\}$ such that

$$\{1 \leqslant x_i \leqslant e_i;\ 1 \leqslant i \leqslant d\} \subset \iota \subset \{1 \leqslant x_i \leqslant E_i;\ 1 \leqslant i \leqslant d\} \tag{34}$$

where $e_i = E_i = l_i(\iota)$ for $i \notin \{1, Q\}$, $e_i + 1 = E_i = l_i(\iota)$ for $i \in \{1, Q\}$, and both inclusions are strict.

The last step in the proof of the theorem is to transform ι into a configuration with at most one face eroded.

Suppose that the element of $\widetilde{\mathscr{B}^{d-1}}$ chosen on the subspace $\{x_1 = l_1(\iota)\}$, $\iota_{l_1(\zeta)}$, has the length along direction Q that is smaller than $l_Q(\iota)$. In this case $\iota_{l_1(\iota)}$ and $\iota \cap \{x_Q = l_Q(\iota)\}$ are disjoint and we have

$$PA^d(\iota) = PA^d(\chi) + PA^{d-1}(\iota_{l_1(\iota)}) + PA^{d-1}(\iota \cap \{x_Q = l_Q(\iota)\}) \tag{35}$$

where $\chi = \{1 \leqslant x_i \leqslant e_i;\ 1 \leqslant i \leqslant d\}$ as in (34).

To verify (35) note that the lines that could contribute to both $(d-1)$-dimensional projected areas in the right-hand side while contributing only once to $PA^d(\iota)$ would have to be subset of $\{x_1 = l_1(\iota),\ x_Q = l_Q(\iota)\}$, which is disjoint from ι.

We then apply Lemmas 6 and 7 to $\iota_{l_1(\iota)}$ and $\iota \cap \{x_Q = l_Q(\iota)\}$, increasing one and decreasing the other until the largest one, say the one on $\{x_1 = l_1(\iota)\}$, which is in $\widetilde{\mathscr{B}^{d-1}}$, has its $(d-1)$-dimensional block equal to the face of χ [χ defined after Eq. (35)]. The resulting configuration is in $\mathscr{B}^d$ and Lemma 3 is proven in this case.

The last possibility to be considered is that $\iota_{l_1(\iota)}$ and $\iota \cap \{x_Q = l_Q(\iota)\}$ are not disjoint. In this case $\iota_{l_1(\iota)}$ must have its $(d-1)$-dimensional block equal to a face of χ and ι is already in $\mathscr{B}^d$. This finishes the proof of Lemma 2 and therefore of Theorem 3.

We now give the proof of Theorem 2.

For part (a) take $\alpha \in \mathscr{C}^{\eta,+}(c)$ with $c = |\mathbf{p}_{d-1}|$. Then

$$PA^d(\alpha) \geqslant PA^d(\alpha \cap \{x_1 \leqslant l_1(\eta)\}) + \sum_{j \geqslant l_1(\eta)+1} PA^{d-1}(\alpha \cap \{x_1 = j\}) \qquad (36)$$

To check this, note that a line along direction 1 which intersects $\alpha \cap \{x_1 \geqslant l_1(\eta)+1\}$ but not $\alpha \cap \{x_1 \leqslant l_1(\eta)\}$ contributes only to the left-hand side of (36).

By Lemma 5 and Theorem 3 the second therm in the right-hand side of (36) is larger than or equal to $p_{c_1}^{d-1}$ [as in (22)] with $c_1 = |\alpha \cap \{x_1 \geqslant l_1(\eta)+1\}|$. Now, $p_{c_1}^{d-1}$ is the $(d-1)$-dimensional area of an element of $\mathscr{B}^{d-1}(c-1)$, say $\bar{\alpha}_1$, that, by Lemma 4, we may choose so that

$$PA^d(\alpha) \geqslant PA^d((\alpha \cap \{x_1 \leqslant l_1(\eta)\}) \cup \bar{\alpha}_1) \qquad (37)$$

The same arguments give

$$PA^d(\alpha) \geqslant PA^d((\alpha \cap \{x_1 \leqslant l_1(\eta)\} \cap \{x_1 \geqslant 1\}) \cup \bar{\alpha}_1 \cup \bar{\alpha}_2) \qquad (38)$$

with $\bar{\alpha}_2$ conveniently chosen in $\mathscr{B}^{d-1}(c_2)$ with $c_2 = |\alpha \cap \{x_1 \leqslant 0\}|$.

Doing the same along all lattice directions, we finally get

$$PA^d(\alpha) \geqslant PA^d\left(\eta \bigcup_{i=1}^{2d} \bar{\alpha}_i\right) \geqslant PA^d(\eta \cup \bar{\alpha}) \qquad (39)$$

where $\bar{\alpha}_i \in \mathscr{B}^{d-1}(c_i)$ with $c_i = |\alpha \cap \{x_i \geqslant l_i(\eta)+1\}|$ if i is odd and $c_i = |\alpha \cap \{x_i \leqslant 0\}|$ if i is even and $\bar{\alpha} \in \mathscr{B}^{d-1}(c)$. The last inequality holds again by Lemma 4 and Theorem 3. We may chose $\bar{\alpha}$ in $\{x_d = l_d(\eta)\}$ so that $\eta \cup \bar{\alpha} = \bar{\eta}$ and part (a) is proven.

For part (b) take now $\alpha \in \mathscr{C}^{\eta,-}(c)$. Follow the arguments in the proof of Theorem 3 to get ζ and transform it as done in Transformation 2. With this we get to a configuration on which all missing sites as compared to η are in $\eta \cap \{x_d = l_d(\eta)\}$. By choosing the $(d-1)$-dimensional configuration there in $\mathscr{B}^{d-1}(c)$ we may get $\underline{\eta}$ and part (b) is proven.

To verify part (c), first note that the same proofs given for (a) and (b) hold if instead of c one adds or removes a smaller number k of sites with $\bar{\eta}$ and $\underline{\eta}$ replaced by the corresponding configurations with $\mathbf{p}_{d-1}$ replaced by an element $\rho_k \in \mathscr{B}^{d-1}(k)$. This element is a union of blocks as in (17). Each i-dimensional block in this decomposition has sides which are smaller than $L(i)$, the smallest integer larger than $2(i-1)/h$, since $k \leqslant c = \sum_{j=1}^{d} L(j)^{j-1} [L(i)-1]$. We finish the proof of (c) by verifying that each of these blocks have positive energy, which is a simple computation.

ACKNOWLEDGMENTS

This problem originated from a collaboration with R. H. Schonmann. I am especially grateful to him for his advice and encouragement. I am also grateful to several colleagues with whom this problem was discussed, among them M. Cassandro, R. Durrett, P. Ferrari, L. R. Fontes, Y. Kohayakawa, E. Olivieri, and P. Picco. This work was partially supported by CNPq and FAPESP.

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